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SOLVING SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH PROBABILISTIC POTENTIAL THEORY

BY

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ABSTRACT. Techniques of probabilistic potential theory are applied to solve $-Lu + f(u) = \mu$, where μ is a signed measure, f a (possibly discontinuous) function and L a second order elliptic or parabolic operator on R^d or, more generally, the infinitesimal generator of a Markov process. Also formulated are sufficient conditions guaranteeing existence of a solution to a countably infinite system of such equations.

1. Introduction. Our main aim in this article is to formulate conditions guaranteeing existence of solutions to the semilinear equation

$$-Au + f(u) = \mu, \tag{1.1}$$

where A is a second order elliptic or parabolic operator on R^d or some open domain in R^d, μ is a signed measure and the nonlinear term f satisfies certain growth conditions detailed below. Whereas current practice in solving (1.1) relies mostly on standard partial differential equations techniques (such as Sobolev spaces and L^p-convergence), we reformulate the problem as one in probabilistic potential theory. There are several advantages to this approach, not the least of which is that it is no harder to treat by the same method the case where A is the infinitesimal generator of a Markov process. In this generality, $-A$ may be an integro-differential operator (such as $(-\Delta)^\alpha$, $0 < \alpha < 1$), and the underlying space E may be more general than R^d (such as a manifold, or even just a Borel subset of a compact metric space).

Let P_t be a sub-Markov semigroup on E of a Markov process X_t with infinitesimal generator $A = \lim_{t \downarrow 0} t^{-1}(P_t - I)$. The basic ideas of Markov process theory are recalled briefly in the Appendix. We assume throughout that X has a dual right continuous strong Markov process $\hat{X}$ on E (which corresponds, roughly, to assuming that A^*, the adjoint of A, is the infinitesimal generator of $\hat{P}_t$ and $\hat{X}_t$) (see §2). Let $U = (-A)^{-1}$ and $u(x, y)$ be the Green operator (or potential) and the Green function so that $U(x, dy) = u(x, y)m(dy)$. Instead of solving (1.1), we solve

$$-Au + \gamma(x)u = \mu + \gamma(x)u - f(u)$$

or

$$u = V[\mu + \gamma(x)u - f(u)], \tag{1.2}$$

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where $V = [-A + \gamma]^{-1}$ and $\gamma \geq 1$ is a function chosen so that $u \to \gamma(x)u - f(u)$ is increasing on $[-U|\mu|(x), U|\mu|(x)]$ whenever $U|\mu|(x) < \infty$. The construction of $V(x, dy) = v(x, y)m(dy)$ requires some care. A precise version of v is needed, and γ may be so large that $E - F = F^c = \{x\colon v(x, \cdot) \equiv 0\} \neq \varnothing$. Probabilists may be interested in the necessity of using such large γ for relatively simple cases of (1.1). Fortunately in Markov process theory, it has long been known how to choose a precise version of V (see §2). The hypotheses required on f are: (i) γ can be chosen as described above; (ii) f is either right or left continuous; (iii) $zf(z) \geq 0$. If f has one continuous derivative, then (i) and (ii) are automatically satisfied. In the literature on the subject (which we discuss later in the introduction), nonlinearities $f(x, u)$ involving x are often permitted. Minor modifications of our proofs allow us to treat this case also, but we leave this to the interested reader.

If $U|\mu| < \infty$ m almost everywhere (abbreviated m-a.e.) and if f satisfies (i)–(iii) above, then we show (Theorem (2.13)) that (1.2) always has a solution u on F and $-U\mu^- \leq u \leq U\mu^+$. We do this by setting $u_0 = U\mu^+$, $v_0 = -U\mu^-$ and iterating:

$$u_{n+1} = V[\mu + \gamma u_n - f(u_n)], \qquad v_{n+1} = V[\mu + \gamma v_n - f(v_n)].$$

We show that u_n decreases while v_n increases. This method is a refinement of the well-known method of sub- and super-solutions (see e.g. [**15**, **18**]). Once we have a solution to (1.2), we need to show that it satisfies (1.1). (The solution to (1.2) is useful as it stands in the potential theory and the probability theory: this seems to be a worthwhile topic for future study.) Let $(h, k) = \int hk\, dm$. In §3, we show that if A is a second order elliptic or parabolic operator on R^d, and if $1_F\gamma \cdot U|\mu|$ is locally integrable, then u solves $-Au + f(u) = 1_{\hat{F}} \cdot \mu$ (note: $\hat{F}^c = \{y\colon v(\cdot, y) = 0\}$; see the discussion following (2.9)) in the sense that $\mu(1_{\hat{F}}h) = (-A^*h, u1_{\hat{F}}) + (h, f(u)1_{\hat{F}})$ for every function h which is infinitely differentiable with compact support (Theorem (3.11)). A natural condition for u to solve (1.1) on all of R^d is given in Corollary (3.13): $\gamma \cdot U|\mu|$ should be locally integrable, and for each $x \in R^d$, $u(\cdot, x)\gamma(\cdot)$ should be locally integrable. These results extend immediately to smooth manifolds. There are natural situations in which $F \neq R^d$. In fact, we discuss an example in which $F = R^3 - \{0\}$, which occurs when attempting to solve the equation $-\frac{1}{2}\Delta u + u^3 = \varepsilon_0$.

As we mentioned before, we feel there are several advantages to this approach. The first has already been mentioned: infinitesimal generators of general Markov processes having strong duals are admitted. Nor do the infinitesimal generators need to be selfadjoint or very smooth, and the space E need not have smooth boundary (or any boundary at all). The second appealing feature of this approach is that, modulo the perhaps unfamiliar probabilistic potential theory used, the computations are elementary. By converting to a problem in potential theory, we get a sequence of iterates converging monotonically to a solution, and the most difficult result we need to apply is the monotone convergence theorem to obtain the solution to the potential theory formulation (1.2). To show that the solution to (1.2) solves (1.1) is also straightforward under the hypotheses discussed above (see §3). The most complicated result we use there is Ito's change of variables formula, although there are undoubtedly alternate analytic methods which might be used. It is worth noting that most applications of the sub- and super-solution method need to use strong regularity estimates (see e.g. [**15**], where D. Sattinger appeals to the results of [**1**]). The third attractive facet of the approach is the introduction of the precise set F on which (1.1) is solved (see [**3**] for an idea in a similar spirit). This appears as a natural and necessary ingredient in the probabilistic approach to the problem. Finally, while probability theory has often successfully interacted with linear partial differential equations, fewer connections have been made between Markov process theory and nonlinear partial differential equations. We hope that this article demonstrates the potential for fruitful interaction between these two subjects.

§2 is devoted to solving (1.2) for transient Markov processes (i.e. $u(x, y) < \infty$ m-a.e.); §4 treats the nontransient case. In §3, we show that the solution to (1.2) solves (1.1) for various elliptic and parabolic operators. The hypotheses on f leave out nonlinearities such as $f(u) = u^{1/3} \wedge 1 \vee (-1)$. In §5, we take a look at solving (1.1) for this type of nonlinearity. We show that the following *countably infinite* system of equations can be solved on $D \subset R^d$:

$$\begin{array}{ccc} (-\Delta + \alpha)u_1 - f_1(u_1, u_2, \ldots) = \mu_1 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ (-\Delta + \alpha)u_n - f_n(u_1, u_2, \ldots) = \mu_n \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{array}$$

where $U|\mu_i| < \infty$ m-a.e., and (f_i) is a uniformly bounded sequence of continuous functions, each mapping R^∞ into R^1. If D is bounded, $\alpha \geqslant 0$, and if $D = R^d$, $\alpha > 0$. An Appendix has been added to serve as orientation for analysts and probabilists in Markov process theory: the reader is encouraged to at least skim through it.

This introduction would not be complete without discussing at least briefly how these results relate to the existing (enormous) literature on semilinear equations (see e.g. [**2, 4, 6, 7, 14, 15, 18**]). The state of the art is perhaps best summarized in a recent preprint of Brezis [**6**]. We recall several of his results below for comparison. Our techniques are quite different from those used in [**6**]. In fact, one can specify boundary limits at infinity for solutions of equations $-\Delta u + f(u) = \mu$ in R^d, but the limit must be taken in the sense of the fine topology on R^d generated by the Brownian motion (see the discussion at the end of §3).

(1.3) *Monotone nonlinearities*. Let $1 < p < \infty$. For every $f \in L^1_{\text{loc}}(R^N)$, there is a unique $u \in L^p_{\text{loc}}(R^N)$ so that $-\Delta u + |u|^{p-1}u = f(x)$. Brezis states that $|u|^{p-1}u$ may be replaced above by a C^1 function g with $g'(u) \geqslant a|u|^{p-1}$.

(1.4) *Nonmonotone nonlinearities*. Let $g(x, u)$ be measurable in x and continuous in u so that $g(x, u)\operatorname{sgn}(u) \geqslant a|u|^p - w(x)$, where $w \in L^1_{\text{loc}}(R^N)$, $a > 0$, $1 < p < \infty$ and

$$h_M(x) = \sup_{|u| \leqslant M} |g(x, u)| \in L^1_{\text{loc}}(R^N) \quad \forall M > 0.$$

Then there is a $u \in L^p_{\text{loc}}(R^N)$ so that $g(\cdot, u) \in L^1_{\text{loc}}(R^N)$ and $-\Delta u + g(x, u) = 0$.

(1.5) *Measures on the right*. Let $\mu = f + \Delta\phi$, where $f \in L^1_{\text{loc}}(R^N)$ and $\phi \in L^p_{\text{loc}}(R^N)$. Then $-\Delta u + |u|^{p-1}u = \mu$ has a unique solution in $L^p_{\text{loc}}(R^N)$ $(1 < p < \infty)$.

Undoubtedly, (1.3)–(1.5) can be combined somewhat to yield a more general result. But it is clear that these results all relate the growth of f to polynomial growth and the measure μ in (1.5) to L^P (see also [3]). Of course, with these conditions, the fact that $u \in L^P_{\text{loc}}$ is also obtained. But these results would not include $g(u) = \text{Arctan}(u)$, nor would they include a nonlinearity with a discontinuity. Our philosophy is that given the nonlinearity f, we delineate the class of measures for which (1.1) can be solved.

Finally, let us set out the notations and conventions we use. If E is a metric space, $C(E)$ (resp. $bC(E)$) will denote the continuous functions on E (resp. which are bounded). If $\mathfrak{E}$ is a σ-algebra, we will use the same symbol $\mathfrak{E}$ (resp. $b\mathfrak{E}$, $\mathfrak{E}^+$) to denote the collection of $\mathfrak{E}$-measurable functions (resp. which are bounded, which are positive). In case E is an open set in R^d, $C_K^\infty(E)$ denotes the collection of infinitely differentiable functions with compact support in E, while if $F \subset R^d$, $L^P(F)$ (resp. $L^P_{\text{loc}}(F)$) denotes the set of Borel functions f with $|f|^P$ integrable (resp. locally integrable) with respect to Lebesgue measure.

2. The potential theory formulation. Let E be a Lusin topological space (i.e. E is homeomorphic to a Borel subset of a compact metric space) with Borel field $\mathfrak{E}$. Let $X = (\Omega, \mathfrak{F}, X_t, (P^x)_{x \in E})$ be a transient right continuous strong Markov process on E with a Borel measurable semigroup (P_t) and resolvent (U^α) (see the Appendix for a description of these objects). Suppose that P_t has infinitesimal generator $A = \lim_{t \downarrow 0} t^{-1}(P_t - I)$. Our purpose in this section is to reformulate the following problem in terms of probabilistic potential theory and to solve it:

Problem. Given f: $R \to R$ and a measure μ on E, find a function u satisfying $-Au + f(u) = \mu$.

In order to do this, we need some assumptions. We assume the following regularity hypothesis: there is another right continuous strong Markov process $\hat{X} = (\hat{\Omega}, \hat{\mathfrak{F}}, \hat{X}_t, (\hat{P}^x)_{x \in E})$ which is in duality with X with respect to a σ-finite measure m on E. Crudely put, we are requiring that the "adjoint" semigroup be associated with a nice Markov process. That is, if $(\hat{P}_t)$ and $(\hat{U}^\alpha)$ are the semigroup and resolvent of $\hat{X}$, then

$$\begin{aligned} &\int_E f \cdot P_t g \, dm = \int_E g \cdot \hat{P}_t f \, dm \quad \text{for every } f, g \in \mathfrak{E}^+; \\ &U^\alpha(x, \cdot) \ll m \quad \text{and} \quad \hat{U}^\alpha(y, \cdot) \ll m. \end{aligned} \tag{2.1}$$

The process $\hat{X}$ will not figure explicitly in our results, but its existence implies several useful consequences (see Chapter VI of [5]). For each $\alpha \geqslant 0$, there is a function $u^\alpha(x, y) \in \mathfrak{E} \times \mathfrak{E}$ so that

$$\begin{aligned} &U^\alpha(x, dy) = u^\alpha(x, y)\, m(dy), \qquad \hat{U}^\alpha(y, dx) = u^\alpha(x, y)\, m(dx), \\ &\text{for each } y \in E, \quad x \to u^\alpha(x, y) \text{ is } \alpha\text{-excessive for } P_t, \\ &\text{for each } x \in E, \quad y \to u^\alpha(x, y) \text{ is } \alpha\text{-excessive for } \hat{P}_t. \end{aligned} \tag{2.2}$$

If another function satisfies the conditions in (2.2), it must in fact be u^α; $u(x, y) \equiv u^0(x, y)$ is the Green function of A.

(2.3) EXAMPLE. If $E = R^d$, $d \geq 3$, and if X_t is Brownian motion, then we may take $\hat{X}_t$ to be Brownian motion, $A = \Delta/2$, m to be Lebesgue measure on R^d, and

$$u(x, y) = \frac{c(d)}{|x - y|^{d-2}},$$

where $c(d)$ is a constant depending on the dimension d.

If μ is a positive measure on E, we set $U^\alpha\mu(x) = \int u^\alpha(x, y)\,\mu(dy)$. If μ is a signed measure, and if $U^\alpha|\mu| < \infty$ m-a.e., then we set $U^\alpha\mu(x) = U^\alpha\mu^+(x) - U^\alpha\mu^-(x)$ on $\{U^\alpha|\mu| < \infty\}$. We set $U^\alpha\mu(x) = \infty$ on $\{U^\alpha|\mu| = \infty\}$ for convenience and abbreviate $U^0\mu$ as $U\mu$.

Now we come to the hypotheses on the nonlinearity f.

Let $f\colon R \to R$ be a function so that

(2.4) f is either right continuous or left continuous,

(2.5) $zf(z) \geq 0$,

(2.6) for every $0 \leq z < \infty$, there is a number $\rho(z) \geq 1$
so that $\rho(z)u - f(u)$ increases (as a function of u) on $[-z, z]$.

Condition (2.5) appears standard in the literature (see e.g. **[6, 7]**). We extend ρ so that $\rho(\infty) = \infty$. Note that (2.6) is automatically satisfied if f is continuously differentiable: take $\rho(z) = \sup\{1 + |f'(r)|\colon -z \leq r \leq z\}$. Condition (2.6) forces $f(u+) - f(u-) \leq 0$ for every u; i.e. f can jump down only. This fact together with (2.5) implies $f(0+) = f(0-) = f(0) = 0$. Now fix a signed measure μ on E so that $U|\mu| < \infty$ m-a.e., and set $\gamma(x) = \rho(U|\mu|(x))$. Set $\Gamma(x) = [-U|\mu|(x), U|\mu|(x)]$ if $U|\mu|(x) < \infty$ and $\Gamma(x) = \varnothing$ if $U|\mu|(x) = \infty$. Then $\gamma(x)u - f(u)$ increases in u on $\Gamma(x)$.

We now use γ to "kill" X and $\hat{X}$ in the following way (see Chapter III of **[5]** for a general exposition on killing Markov processes with multiplicative functionals). Set

(2.7) $$A_t = \int_0^t \gamma(X_s)\,ds, \quad T = \inf\{t > 0\colon A_t = \infty\}, \quad M_t = \exp(-A_t)1_{[0,T)}(t).$$

A_t is an additive functional and M_t is a multiplicative functional. Define

(2.8) $$Q_t g(x) = E^x(g(X_t)M_t),$$
$$V^\alpha g(x) = \int e^{-\alpha t} Q_t g(x)\,dt; \qquad Vg(x) = V^0 g(x).$$

That Q_t is a semigroup can be verified by using the Markov property (see **[5]**). Formally, Q_t is the semigroup with infinitesimal generator $A - \gamma$, and $(-A + \gamma)^{-1} = V$. But this is actually a delicate point in our discussion, since γ has no nice smoothness properties and is unbounded in general. We justify the statement for a class of elliptic operators in §3 under various hypotheses. It is this justification which is needed to show that the "potential theory solution" (2.13) of the Problem (P) actually solves it. So far, we have given the analytic description of "killing". One can interpret this probabilistically by showing that Q_t is the semigroup of a Markov process obtained from X by terminating the life of X at a random time and sending it to a "cemetery point". We do not need this probabilistic description, so we do not formalize it here.

It is easy to see that it can happen that for some $x \in E$, $Q_t 1(x) = 0$ for all $t \geqslant 0$. Let

(2.9) $$F = \{x \in E: V1(x) > 0\} = \{x \in E: P^x(T > 0) = 1\},$$

F is called the set of permanent points of M. In general, $F^c \neq \varnothing$. What this means is that if $x \in F^c$, $M_t = 0$ for all $t \geqslant 0$; i.e. the multiplicative functional "kills" X_t immediately at time 0. It is evident that, in some sense, V^α should be unable to give us information about what is going on in F^c. We make this idea precise below.

EXAMPLE ((2.3) CONTINUED). The situation where $F^c \neq \varnothing$ shows up quite quickly in simple problems. If we try to solve $-\frac{1}{2}\Delta u + u^3 = \varepsilon_0$ on R^3, then we may take $\rho(z) = 1 + 3z^2$ and $\gamma(x) = 1 + 3(U\varepsilon_0(x))^2$. We claim $F^c = \{0\}$. (The argument is Dellacherie's [**9**].) By the law of the iterated logarithm [**13**] (recall $U\varepsilon_0(X_t) = c(3)\|X_t\|^{-1}$),

$$P^0\left[\limsup_{t \downarrow 0} \frac{\|X_t\|^2}{2t\log(\log 1/t)} = 1\right] = 1.$$

Inverting, we have

$$\liminf_{t \downarrow 0} \|X_t\|^{-2} 2t \log\left(\log\frac{1}{t}\right) = 1 \quad \text{a.s.}\ (P^0).$$

Thus, for P^0-almost all ω, there is a $\delta(\omega) > 0$ so that

$$\|X_t\|^{-2} > \left(4t\log\left(\log\frac{1}{t}\right)\right)^{-1} \quad \text{if } t < \delta(\omega).$$

Since $t \to (4t\log(\log\frac{1}{t}))^{-1}$ is not integrable in a neighborhood of zero, it follows that $P^0(A_{0+} = \infty) = 1$. Since $\gamma(x)$ is continuous away from 0, and $P^x(X_t \text{ hits } 0) = 0$ for every $x \neq 0$, $A_t < \infty$ a.s. P^x for every $x \neq 0$ for every t.

In the same way as described above for X, one can "kill" $\hat{X}$ with a dual multiplicative functional by setting:

$$\hat{A}_t = \int_0^t \gamma(\hat{X}_s)\, ds, \qquad \hat{T} = \inf\{t > 0: \hat{A}_t = \infty\},$$

$$\hat{M}_t = \exp(-\hat{A}_t)1_{[0,\hat{T})}(t), \qquad \hat{Q}_t g(x) = \hat{E}^x(g(\hat{X}_t)\hat{M}_t),$$

$$\hat{V}^\alpha g(x) = \int e^{-\alpha t}\hat{Q}_t g(x)\, dt, \qquad \hat{F} = \{x \in E: \hat{V}1(x) > 0\}.$$

Then the semigroups Q_t and $\hat{Q}_t$ are in duality:

$$\int_E f \cdot Q_t g\, dm = \int_E g \cdot \hat{Q}_t f\, dm,$$

$$V^\alpha(x, \cdot) \ll m \quad \text{and} \quad \hat{V}^\alpha(y, \cdot) \ll m.$$

In general, $F \neq \hat{F}$, although $m(F - \hat{F}) = m(\hat{F} - F) = 0$. The functionals M_t and $\hat{M}_t$ are examples of dual exact multiplicative functionals. Such objects have been thoroughly studied in Getoor [**12**], and we recall some facts we need. If we set

$$P_M f(x) = \begin{cases} -E^x \int_0^t f(X_t)\, dM_t & \text{if } x \in F, \\ f(x) & \text{if } x \in E - F, \end{cases}$$

then there is a function $v(x, y)$ on $E \times E$ so that

$$Vf(x) = \int v(x, y) f(y) m(dy), \qquad \hat{V}f(y) = \int v(x, y) f(x) m(dx),$$

(2.10) for each $y \in E$, $x \to v(x, y)$ is excessive for Q_t,

for each $x \in E$, $y \to v(x, y)$ is excessive for $\hat{Q}_t$.

Note that $v(x, y) = 0$ if $x \in F^c$ or if $y \in \hat{F}^c$. Moreover,

$$u(x, y) = v(x, y) + P_M u(x, y), \tag{2.11}$$

where $P_M u(x, y) = P_M[u(\cdot, y)](x)$. Thus, if $x \in F$,

$$\begin{aligned} P_M u(x, y) &= -E^x \int u(X_t, y)\, dM_t = E^x \int u(X_t, y) \gamma(X_t) M_t\, dt \\ &= \int Q_t[u(\cdot, y)\gamma(\cdot)](x)\, dt = \int v(x, z) u(z, y) \gamma(z) m(dz). \end{aligned} \tag{2.12}$$

As before, if μ is a measure on E, we let $V\mu(x) = \int v(x, y)\mu(dy)$. Note that if f is a function on E, $Vf(x) = \int v(x, y) f(y) m(dy)$.

(2.13) THEOREM. *Let f: $R \to R$ be a function satisfying* (2.4), (2.5) *and* (2.6), *let μ be a signed measure on E with $U|\mu| < \infty$ m-a.e., and construct γ and V as described above. Then there is a function u on F so that $u = V[\gamma u - f(u) + \mu]$ on $\{U|\mu| < \infty\}$. Moreover, $-U\mu^- \leqslant u \leqslant U\mu^+$ on $\{U|\mu| < \infty\}$.*

PROOF. Let $\Lambda = \{U|\mu| < \infty\}$. Recall $m(E - \Lambda) = 0$. If w is any function on E, define a new function H by setting $H(w)(x) = \gamma(x)w(x) - f(w(x))$, and set $u_0 = U\mu^+$ and $v_0 = -U\mu^-$. Integrating (2.11) with respect to μ^+, we obtain $u_0 = V\mu^+ + P_M u_0$. Using (2.12), we have $u_0 = V\mu^+ + V(u_0\gamma)$ on F. Similarly, we obtain $v_0 = -V\mu^- + V(v_0\gamma)$ on F. Now set

$$\begin{aligned} u_1 &= V[H(u_0) + \mu] \quad \text{on } \Lambda, \\ v_1 &= V[H(v_0) + \mu] \quad \text{on } \Lambda. \end{aligned} \tag{2.14}$$

(2.15) PROPOSITION. (i) *u_1 and v_1 are well defined.*

(ii) *$u_0 \geqslant u_1 \geqslant v_1 \geqslant v_0$ on Λ.*

PROOF. (i) Since $0 \leqslant u_0 \in \Gamma$, $H(u_0) \geqslant -f(0) = 0$ on Λ by our choice of γ. Since V is a positive measure, $V[H(u_0)]$ is well defined. (It may be infinity.) Since $v(x, y) \leqslant u(x, y)$, $V\mu$ is well defined and finite on Λ. Therefore, u_1 is well defined on Λ (although we need to rule out the possibility that u_1 may be infinite). Similarly, one obtains v_1 is well defined on Λ (although we need to rule out the possibility that v_1 may be negative infinity).

(ii) Since $u_0 < \infty$ on Λ, we may compute

$$u_1 - u_0 = V[H(u_0) + \mu - \mu^+ - \gamma u_0] = V[-f(u_0) - \mu^-] \quad \text{on } \Lambda.$$

Since $-f(u_0) - \mu^- \leqslant 0$, we conclude $u_1 \leqslant u_0$ on Λ. Similarly,

$$v_1 - v_0 = V[H(v_0) + \mu + \mu^- - \gamma v_0] = V[-f(v_0) + \mu^+] \quad \text{on } \Lambda.$$

Since $-f(v_0) + \mu^+ \geqslant 0$, we conclude $v_1 \geqslant v_0$ on Λ. Moreover,

$$u_1 - v_1 = V\left[H(u_0) - H(v_0)\right].$$

Since u_0 and v_0 are in Γ, $H(u_0) \geqslant H(v_0)$, so $u_1 - v_1 \geqslant 0$ on Λ. Q.E.D.

We set up the induction step as follows:

$$u_{n+1} = V\left[H(u_n) + \mu\right] \quad \text{on } \Lambda, \tag{2.16}$$
$$v_{n+1} = V\left[H(v_n) + \mu\right] \quad \text{on } \Lambda.$$

Our induction hypotheses are the following:

(2.17) $V[H(u_k) + \mu]$ and $V[H(v_k) + \mu]$ are well defined for every $k \leqslant n - 1$ on Λ.

$$u_0 \geqslant u_1 \geqslant \cdots \geqslant u_n \geqslant v_n \geqslant \cdots \geqslant v_1 \geqslant v_0 \quad \text{on } \Lambda. \tag{2.18}$$

(2.19) PROPOSITION. *Under the assumptions given above,*
(i) $V[H(u_n) + \mu]$ *and* $V[H(v_n) + \mu]$ *are well defined on* Λ.
(ii) $u_0 \geqslant u_1 \geqslant \cdots \geqslant u_{n+1} \geqslant v_{n+1} \geqslant \cdots \geqslant v_1 \geqslant v_0$ *on* Λ.

PROOF. It should come as no surprise that this proof is similar to the one given in (2.15).

(i) Since $u_0 \geqslant u_n \geqslant v_n \geqslant v_0$ on Λ, $H(u_n) \geqslant H(v_0) = \gamma v_0 - f(v_0)$, and $H(v_n) \leqslant H(u_0) = \gamma u_0 - f(u_0)$ on Λ.

Now

$$V\left[\gamma |v_0|\right](x) \leqslant V[\gamma \cdot U|\mu|](x) \leqslant U|\mu|(x) < \infty \quad \text{on } \Lambda$$

since $U|\mu| = V|\mu| + V[\gamma \cdot U|\mu|]$. Since $-f(v_0) \geqslant 0$, $V[\gamma v_0 - f(v_0)]$ is well defined, so $V[H(u_n)]$ is well defined on Λ (although it may be infinity). Similarly, one obtains $V[H(v_n)]$ is well defined on Λ (although it may be negative infinity). Since $u(x, y) \geqslant v(x, y)$, $V\mu$ is well defined and finite on Λ, so we obtain (i).

(ii) On Λ, $u_{n+1} - u_n = V[H(u_n) - H(u_{n-1})]$. By (2.18) and our choice of γ, the term in brackets is negative m-a.e., so $u_{n+1} - u_n \leqslant 0$ on Λ. Similarly, one obtains $v_{n+1} - v_n \geqslant 0$ on Λ. Finally, on Λ, $u_{n+1} - v_{n+1} = V[H(u_n) - H(v_n)]$, and the term in brackets is positive by (2.18). Thus $u_{n+1} \geqslant v_{n+1}$ on Λ. Q.E.D.

This completes the induction step, and we can now complete the proof of (2.13). The sequence u_n decreases to a function u while v_n increases to a function v on Λ. If f is right continuous, $H(u_n)$ decreases to $H(u)$, while if f is left continuous, $H(v_n)$ increases to $H(v)$. In the first case, the monotone convergence theorem applies to yield $u = V[H(u) + \mu]$, while in the second case, it yields $v = V[H(v) + \mu]$. Q.E.D.

3. The differential equation formulation. In this section, we assume $E = R^d$, and we give sufficient conditions for the solution u obtained in (2.13) to solve the Problem stated at the beginning of §2 when A is either an elliptic or parabolic partial differential operator. One advantage of using the Markov approach to these problems is that there is no essential difference in the treatment of these two cases.

Consider the following elliptic operator $\mathfrak{L}$ defined by

$$\mathfrak{L}f(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x) D_i D_j f(x) + \sum_i b_i(x) D_i f(x),$$

where

- $D_i f(x) = \partial f(x)/\partial x_i$;
- for each $x \in R^d$, $a(x) = (a_{ij}(x))$ is a $d \times d$ symmetric positive matrix;
- $a_{ij} \in bC^2(R^d)$ and a^{-1} exists;
- $b(x) = (b_i(x))$ is a vector of bounded measurable functions on R^d.

Then $\mathfrak{L}$ is the infinitesimal generator of a Markov process (Y_t, R^x) with continuous trajectories on R^d in the following sense:

(3.1) $R^x(Y_0 = x) = 1$;

(3.2) for every $f \in C_K^\infty(R^d)$, $f(Y_t) - f(Y_0) - \int_0^t \mathfrak{L}f(Y_s)\,ds$ is a continuous $(\mathfrak{F}_t^0)$-martingale, where $\mathfrak{F}_t^0 = \sigma(Y_s\colon s \leqslant t)$.

Conditions (3.1) and (3.2) characterize the law of (Y_t, R^x). Such a process may be constructed using stochastic differential equations or the martingale problem formulation [**17**]. Moreover, the hypotheses imply that there is a process $\hat{Y}$ in duality with Y with respect to Lebesgue measure on R^d. Not surprisingly, $\hat{Y}$ has infinitesimal generator $\mathfrak{L}^*$, the adjoint of $\mathfrak{L}$ (in the sense of (3.1) and (3.2)).

Now let $\overline{\mathfrak{L}}$ be the parabolic operator on R^{1+d} defined by

$$\overline{\mathfrak{L}}g(t,x) = \frac{\partial g}{\partial t}(t,x) + \mathfrak{L}g(t,x).$$

It is easy to construct the process $(\overline{Y}_t, \overline{R}^{s,x})$ with infinitesimal generator $\overline{\mathfrak{L}}$ (in the sense of (3.1) and (3.2)) as follows. Let (Z_t, Q^s) be the process on R^1 so that $Q^s(Z_t = s + t) = 1$. Then $\overline{Y}_t = (Z_t, Y_t)$ and $\overline{R}^{s,x} = Q^s \times R^x$. $\overline{Y}_t$ has a dual as discussed in §2.

Let us now return to the discussion and notation of §2. In what follows, *the process* (X_t, P^x, L) *is taken to be either* $(Y_t, R^x, \mathfrak{L})$ *on* R^d *or* $(\overline{Y}_t, \overline{R}^x, \overline{\mathfrak{L}})$ *on* $R^{1+(d-1)} = R^d$, so we are discussing the elliptic and parabolic cases simultaneously. The following result is a piece of "folklore", although we could not find a precise statement in the literature, much less a proof. Since we need the result, we prove it here. It formalizes the idea that $(-L+\gamma)^{-1} = V$. The novelty of the result is found in the fact that γ may be unbounded; in fact, so large that F may not be all of R^d. Let $L^*(\hat{F}) = (L^1(\hat{F}) \cap L^2(\hat{F}) \cap L^\infty(\hat{F}))^+$; $k \in L^*(\hat{F})$ is extended to be zero on $\hat{F}^c$.

(3.3) THEOREM. *Assume the notation and hypotheses of* (2.13). *If* $h \in C_K^\infty(R^d)$ *and if* k *is a finely continuous function for* $\hat{Q}_t$ *on* $\hat{F}$ *which is in* $L^*(\hat{F})$, *then*

$$-\int kh\,dm = \int k \cdot V[Lh - \gamma h]\,dm.$$

Recall that the fine topology for $\hat{Q}_t$ is the weakest topology on $\hat{F}$ making the functions $\{\hat{V}\mu\colon \mu \text{ is a finite positive measure}\}$ continuous. If k and h are two functions, we let $(k,h) = \int kh\,dm$. The following elementary result will justify using the Lebesgue dominated convergence theorem at various points in the proof.

(3.4) LEMMA. *Under the hypotheses of* (3.3), $V[|Lh| + |h|\gamma]$ *is a bounded function on* R^d.

PROOF. On F^c, $V(x, \cdot) = 0$, so we need only consider $x \in F$. Since $V(x, \cdot) \leqslant U(x, \cdot)$ and Ug is bounded whenever $g \in C_K(R^d)$, $V|Lh|$ is bounded. Now $V|h|\gamma \leqslant \|h\|_\infty V\gamma$, and $V\gamma(x) = E^x\int_0^\infty \gamma(X_s)e^{-A(s)}\,ds \leqslant 1$. Q.E.D.

PROOF OF (3.3). To begin, let $N_t = h(X_t) - \int_0^t Lh(X_s)\,ds$, and apply Ito's formula to $N_t e^{-A(t)}$:

$$N_t M_t - N_0 = \int_0^t M_s\,dN_s - \int_0^t N_s M_s \gamma(X_s)\,ds,$$

or

$$h(X_t)M_t - h(x) = \int_0^t M_s\,dN_s + M_t\int_0^t Lh(X_s)\,ds + \int_0^t \int_0^s Lh(X_u)\,du\, M_s\gamma(X_s)\,ds - \int_0^t h(X_s)M_s\gamma(X_s)\,ds$$

a.s. (P^x) for $x \in F$. Using integration by parts, we combine the middle two terms on the right-hand side to obtain $\int_0^t M_s Lh(X_s)\,ds$. The first term on the right is an L^2-martingale since M_u is bounded on $[0, s]$ for every s. So in taking the expectation of both sides, the first term on the right becomes zero, and we obtain for each $x \in F$,

$$Q_t h(x) - h(x) = \int_0^t Q_s[Lh - h\gamma](x)\,ds. \tag{3.5}$$

Take $g \in L^*(\hat{F})$. By (3.5),

$$(g, Q_t h - h) = \left(g, \int_0^t Q_s[Lh - h\gamma]\,ds\right) = \int_F \int_0^t \int_{\hat{F}} g(x)[Lh - h\gamma](y)Q_s(x, dy)\,ds\,dx. \tag{3.6}$$

(Recall $(g, h) \equiv \int gh\,dx$.) By (3.4), the absolute value of the integrand on the right-hand side of (3.6) is integrable since $g \in L^1(\hat{F})$. Also note that $(|g|, Q_t|h| + |h|) < \infty$, so we may apply Fubini's theorem to both sides of (3.6) to obtain

$$(\hat{Q}_t g - g, h) = \left(\int_0^t \hat{Q}_s g\,ds, Lh - h\gamma\right).$$

Let $\alpha > 0$ and let $k \in L^*(\hat{F})$ be finely continuous for $\hat{Q}_t$. Then $g = \hat{V}^\alpha k \in L^*(\hat{F})$. Multiply both sides by t^{-1}:

$$\left(t^{-1}[\hat{Q}_t\hat{V}^\alpha k - \hat{V}^\alpha k], h\right) = \left(t^{-1}\int_0^t \hat{Q}_s\hat{V}^\alpha k\,ds, Lh - h\gamma\right). \tag{3.7}$$

Now

$$t^{-1}\int_0^t \hat{Q}_s\hat{V}^\alpha k\,ds \leqslant t^{-1}\int_0^t e^{\alpha s}\hat{V}^\alpha k\,ds = t^{-1}(e^{\alpha t} - 1)\hat{V}^\alpha k \leqslant C\hat{V}^\alpha k$$

for t small. By (3.4), we have $(C\hat{V}^\alpha k, |Lh - h\gamma|) \leqslant (Ck, V^\alpha[|Lh| + |h|\gamma]) < \infty$, so the dominated convergence theorem applies on the right-hand side of (3.7). Since

$\lim_{s\downarrow 0}\hat{Q}_s\hat{V}^\alpha k(x) = \hat{V}^\alpha k(x)$ for every x in $\hat{F}$, we obtain $(\hat{V}^\alpha k,\ Lh - h\gamma)$ upon passing to the limit. As for the left side of (3.7), one can also check that

$$t^{-1}[\hat{Q}_t\hat{V}^\alpha k - \hat{V}^\alpha k] \leqslant t^{-1}(e^{\alpha t} - 1)\hat{V}^\alpha k \leqslant C\hat{V}^\alpha k$$

for t small. Since $t^{-1}[\hat{Q}_t\hat{V}^\alpha k - \hat{V}^\alpha k]$ converges to $\alpha\hat{V}^\alpha k - k$, the dominated convergence theorem applies once again to yield

$$(\alpha\hat{V}^\alpha k - k, h) = (\hat{V}^\alpha k, Lh - h\gamma).$$

As α decreases to zero, $\alpha\hat{V}^\alpha k$ converges boundedly to zero. On the right side, $\hat{V}^\alpha k \leqslant \hat{V}k$ and $(\hat{V}k, |Lh - h\gamma|) < \infty$, so one last application of dominated convergence yields

$$(3.8)\qquad -(k,h) = (\hat{V}k, Lh - h\gamma) = (k, V[Lh - h\gamma]).\quad \text{Q.E.D.}$$

Since (3.8) holds for every $k \in L^*(\hat{F})$ which is finely continuous for $\hat{Q}_t$, (3.8) holds for every $k \in L^*(\hat{F})$. Since $V(x, F^c) = 0$, it follows that for each $h \in C_K^\infty(R^d)$, $V[(Lh - h\gamma)1_F] = -h1_F$ m-a.e. But each function is finely continuous for Q_t on F, so these two functions are in fact equal everywhere. It is the dual, or adjoint version of this we need, so let L^* be the adjoint of the differential operator L.

(3.9) COROLLARY. *Assume the notation and hypotheses of* (2.13). *If* $h \in C_K^\infty(R^d)$, *then*

$$(3.10)\qquad -h1_{\hat{F}} = \hat{V}[(L^*h - h\gamma)1_{\hat{F}}]\quad \text{on } \hat{F}.$$

Now we formulate sufficient conditions for f and μ to solve the problem stated at the beginning of §2.

(3.11) THEOREM. *Assume the hypotheses and notation of* (2.13) *and assume* $\gamma U|\mu|1_F$ *is locally integrable. Then there is a function* u *on* F *so that* $-Lu + f(u) = 1_{\hat{F}}\mu$ *in the sense that* $\mu(1_{\hat{F}}h) = (-L^*h, u1_{\hat{F}}) + (h, f(u)1_{\hat{F}})$ *for every* $h \in C_K^\infty(R^d)$.

PROOF. Let $h \in C_K^\infty(R^d)^+$, and suppose u is the solution obtained in (2.13). We check that $(V|H(u)|, |L^*h| + \gamma h1_F) < \infty$: $H(v_0) \leqslant H(u) \leqslant H(u_0)$, so

$$|H(u)| \leqslant |H(u_0)| + |H(v_0)| \leqslant \gamma \cdot U|\mu| + f(u_0) - f(v_0).$$

But $V[\gamma \cdot U|\mu|] \leqslant U|\mu|$. Since $V[\gamma u_0 - f(u_0) + \mu] \geqslant v_0$, $V[f(u_0)] \leqslant 2U|\mu|$, while $V[\gamma v_0 - f(v_0) + \mu] \leqslant u_0$ implies $-V[f(v_0)] \leqslant 2U|\mu|$. Therefore, $V[|H(u)|] \leqslant 5U|\mu|$. But $(|L^*h|, U|\mu|) < \infty$ since $U|\mu|$ is locally integrable and $(\gamma h, 1_F U|\mu|) < \infty$ by hypothesis. Integrating (3.10), we have

$$-(H(u), h1_{\hat{F}}) = (H(u), \hat{V}[(L^*h - h\gamma)1_{\hat{F}}]).$$

By the computations above, $(V|H(u)|, |L^*h - h\gamma|1_{\hat{F}}) < \infty$, so we may apply Fubini's theorem to obtain

$$(3.12)\qquad -(H(u), h1_{\hat{F}}) = (V[H(u)], (L^*h - h\gamma)1_{\hat{F}}).$$

Integrating (3.10) by μ, we obtain $-\mu(h1_{\hat{F}}) = \mu(\hat{V}[L^*h - h\gamma)1_{\hat{F}}])$. It is easy to check that $|\mu|(\hat{V}[|L^*h - h\gamma|1_{\hat{F}}]) < \infty$, so we may interchange order of integration and add to (3.12) to obtain:

$$-\int h1_{\hat{F}}(x)([\gamma u - f(u)]\,dx + \mu(dx)) = (u, (L^*h - h\gamma)1_{\hat{F}}).$$

Note that $m(h\gamma u 1_{\hat{F}}) < \infty$ since $\gamma \cdot U|\mu| 1_F \in L^1_{\mathrm{loc}}$ by hypothesis. Therefore, the equation above may be rewritten as

$$(u 1_{\hat{F}}, -L^* h) + (f(u) 1_{\hat{F}}, h) = \mu(h 1_{\hat{F}}). \quad \text{Q.E.D.}$$

Undoubtedly, the most interesting special case of (3.11) is when $\hat{F} = R^d$.

(3.13) COROLLARY. *Assume the hypotheses and notation of* (2.13) *and*

(i) $\gamma U|\mu|$ *is locally integrable*,

(ii) *for each* $x \in R^d$, $u(\cdot, x)\gamma(\cdot)$ *is locally integrable.*

Then there is a function u *on* R^d *so that* $-Lu + f(u) = \mu$ *in the sense described in* (3.11) (*with* $\hat{F} = R^d$).

PROOF. As soon as we show $\hat{F} = R^d$, (3.11) will apply to give (3.13). If

$$S = \inf\{t > 0 \colon \hat{X}_t \notin B(r, x)\}$$

(where $B(r, x)$ is the ball of radius r about x), then $\hat{E}^x[\hat{A}_S] \leqslant \hat{U}[1_{B(r,x)}\gamma](x) < \infty$. Thus $\hat{A}_S < \infty$ a.s., so $\hat{F} = R^d$. Q.E.D.

EXAMPLE ((2.3) CONTINUED). We see that if we try to solve $-\frac{1}{2}\Delta u + u^3 = \varepsilon_0$, we obtain a function u on $R^3 - \{0\}$ which satisfies $-\frac{1}{2}\Delta u + u^3 = 0$ in the sense specified above. The third power seems to be the dividing line, however. If we want to solve $-\frac{1}{2}\Delta u + u^{3-\delta} = \varepsilon_0$, we see that $|x - y|^{-1}|y|^{\delta - 2}$ is locally integrable if $\delta > 0$, so the equation can be solved. Note that this condition is (i) in (3.13).

It is worth looking at the boundary behavior near infinity of solutions to the equation $-\Delta u + f(u) = \mu$ on all of R^d. Even in the simple case when $f \equiv 0$, if μ is a measure on R^d which charges every point with rational coordinates, then the solution $u = U\mu$ will equal infinity at every point with rational coordinates. But $\lim_{t\to\infty} U\mu(X_t) = 0$ almost surely (P^x) for every x in R^d. That is, the limit along Brownian paths will be zero (or the fine limit of u at infinity is zero). Recall from (2.13) that $-U\mu^- \leqslant u \leqslant U\mu^+$. Since $\lim_{t\to\infty} - U\mu^-(X_t) = \lim_{t\to\infty} U\mu^+(X_t) = 0$ a.s., $u(X_t) \to 0$ a.s. This remains true if we replace Δ with more general elliptic operators on R^d. The situation on bounded domains is similar.

(3.14) COROLLARY. *Assume the hypotheses of* (3.13). *If* $\mathfrak{L}$ *is an elliptic operator on* R^d ($d \geqslant 3$) *or on a bounded domain* (*with Dirichlet boundary conditions*), *then* $\lim_{t\to\infty} u(X_t) = 0$ *almost surely.*

(3.15) COROLLARY. *Assume the hypotheses of* (3.13). *Then* $f(u) \in L^1_{\mathrm{loc}}(m)$.

PROOF. Since $H(v_0) \leqslant H(u) \leqslant H(u_0)$, $|f(u)| \leqslant |H(v_0)| + |H(u_0)| + 2\gamma|u|$. Since $|u| \leqslant U|\mu|$, we have by following the proof in (3.11) that $V|f(u)| \leqslant 7U|\mu|$. As in (3.11), we obtain

$$-(|f(u)|, h) = (|f(u)|, \hat{V}[(L^*h - h\gamma)]) = (V|f(u)|, L^*h - h\gamma) < \infty. \quad \text{Q.E.D.}$$

4. The nontransient case. In the preceding sections we assumed that X is transient, so $u(x, y) < \infty$ a.e. But elliptic differential operators as discussed in §3 are the infinitesimal generators of nontransient processes when the dimension d equals 1 or 2 (i.e. $u(x, y) \equiv \infty$). Also, if we take $\frac{1}{2}\Delta$ on a bounded domain with Neumann boundary conditions, we obtain a nontransient process which is reflecting Brownian motion. In these cases, $-Lu + g(u) = \mu$ can still be solved if the nonlinearity grows fast enough. Suppose $\alpha > 0$ can be chosen so that

$$(4.1) \quad g(u) - \alpha u \geqslant 0 \quad \text{on } \{u \geqslant 0\} \quad \text{and} \quad g(u) - \alpha u \leqslant 0 \quad \text{on } \{u \leqslant 0\}.$$

We try to solve the problem $-Au + (g(u) - \alpha u) = \mu$, where $A = L - \alpha$. Modifications of previous results are trivial. Let $f(z) = g(z) - \alpha z$.

(4.2) THEOREM. *Suppose g satisfies* (4.1) *and f satisfies* (2.4)–(2.6). *Let μ be a signed measure on R^d with $U^\alpha|\mu| < \infty$, and construct γ and V as described in* §2 (*using $U^\alpha|\mu|$ and $L - \alpha$*). *If*

(i) *for each $x \in R^d$, $u^\alpha(\cdot, x)\gamma(\cdot)$ is locally integrable*;

(ii) *$\gamma U^\alpha|\mu|$ is locally integrable*;

then there is a function u so that $-Lu + f(u) = \mu$.

5. Infinite systems of equations. In §§2 and 3 we assumed that for each $0 \leqslant x < \infty$, there is a number $\rho(x)$ so that $\rho(x)u - f(u)$ is increasing on $[-x, x]$. This requirement rules out some nonlinearities such as $f(u) = u^{1/3} \wedge 1 \vee (-1)$. The technique we discuss now includes such bounded nonlinearities and extends to systems of semilinear equations. In fact, we state the result for a countably infinite system. For simplicity, we shall restrict ourselves to the case of the Laplacian on either R^d or on a bounded domain with Dirichlet boundary conditions. It will be obvious how to extend the results to more general elliptic differential operators and beyond to general infinitesimal generators of Markov semigroups such as fractional Laplacians.

Henceforth, D is a domain in R^d and $u^\alpha(x, y)$ denotes the Green function of $\Delta - \alpha$ on D (with Dirichlet boundary conditions). Also fix:

(5.1) $(\mu_i)_{i \geqslant 1}$ is a sequence of signed measures on D with $U|\mu_i| < \infty$ a.e. for every $i \geqslant 1$;

(5.2) $(f_i)_{i \geqslant 1}$ is a sequence of functions mapping $\overline{R}^\infty$ into R^1 so that if $x_j^n \to x_j$ for every j, then $f(x_1^n, x_2^n, \ldots) \to f(x_1, x_2, \ldots)$.

(5.3) We also require $\sup_{x \in D} \sup_i |f_i(x)| = r < \infty$.

(5.4) THEOREM. *Let D, (μ_i) and (f_i) be as in* (5.1), (5.2) *and* (5.3), *where $\alpha \geqslant 0$ if D is bounded and $\alpha > 0$ if D is R^d. Then we can find a solution to the following countably infinite set of equations*:

$$\begin{aligned} z_1 &= f_1\left[U^\alpha(\mu_i + z_i)_{i \geqslant 1}\right] \\ &\vdots \\ z_n &= f_n\left[U^\alpha(\mu_i + z_i)_{i \geqslant 1}\right] \\ &\vdots \end{aligned}$$

PROOF. Let $\mathbf{N} = \{1, 2, 3, \ldots\}$, and define a measure ξ on $\mathbf{N} \times D$ by setting $\xi(\{i\} \times A) = 2^{-i}\operatorname{leb}(A)$ whenever $A \subset D$, where leb is Lebesgue measure on R^d. We consider $L^\infty(\mathbf{N} \times D)$ as the dual of $L^1(\mathbf{N} \times D, \xi)$ and equip it with the weak-* topology. Thus B_r, the ball of radius r in $L^\infty(\mathbf{N} \times D)$, is compact. Define a transformation T: $L^\infty(\mathbf{N} \times D) \to L^\infty(\mathbf{N} \times D)$ by setting

$$T(g_1, g_2, \ldots) = \left(f_1\left[U^\alpha(\mu_i + g_i)_{i \geqslant 1}\right], f_2\left[U^\alpha(\mu_i + g_i)_{i \geqslant 1}\right], \ldots\right).$$

We claim that T is continuous in the weak-* topology. For if $\bar{g}^n \to \bar{g}$, then $\int \bar{g}^n z \, d\xi \to \int \bar{g} z \, d\xi$ for every $z \in L^1(\mathbf{N} \times D)$. That is, $\sum 2^{-i} \int g_i^n z_i \, dx \to \sum 2^{-i} \int g_i z_i \, dx$. Since $u^\alpha(x_1, \cdot) \in L^1(D, dx)$, $U^\alpha(\mu_i + g_i^n)$ converges to $U^\alpha(\mu_i + g_i)$. By (5.2), $f_j[U^\alpha(\mu_i + g_i^n)_{i \geq 1}]$ converges to $f_j[U^\alpha(\mu_i + g_i)_{i \geq 1}]$ a.e. and boundedly, so

$$\lim_{n \to \infty} \int z \cdot T(\bar{g}^n) \, d\xi = \int z \cdot T(\bar{g}) \, d\xi$$

by the Lebesgue dominated convergence theorem. Thus T is weak-* continuous. Since $L^\infty(\mathbf{N} \times D)$ is locally compact in the weak-* topology, B_r is convex and compact and T: $B_r \to B_r$ (5.3), we know there is a fixed point $\bar{g} \in B_r$ so $T(\bar{g}) = \bar{g}$ by the Schauder theorem. Q.E.D.

(5.5) COROLLARY. *Under the hypotheses of* (5.4), *we can solve the following countably infinite system of equations*:

$$\begin{array}{ccc} (-\Delta + \alpha) u_1 & -f_1(u_1, u_2, \ldots) = & \mu_1 \\ \vdots & & \vdots \\ (-\Delta + \alpha) u_n & -f_n(u_1, u_2, \ldots) = & \mu_n \\ \vdots & & \vdots \end{array}$$

PROOF. By (5.4), the following system has a solution:

$$\begin{array}{ll} z_1 = f_1\left[U^\alpha(\mu_i + z_i)_{i \geq 1}\right] & w_1 = \operatorname{Arctan}\left(U^\alpha(\mu_1 + z_1)\right) \\ \vdots & \vdots \\ z_n = f_n\left[U^\alpha(\mu_i + z_i)_{i \geq 1}\right] & w_n = \operatorname{Arctan}\left(U^\alpha(\mu_n + z_n)\right) \\ \vdots & \vdots \end{array}$$

So

$$\tan w_i = U^\alpha(\mu_i + z_i). \tag{5.6}$$

Thus $z_k = f_k[\tan(w_i)_{i \geq 1}]$. Replacing z_i in (5.6) with this expression, we have

$$\tan(w_i) = U^\alpha\left(\mu_i + f_i\left[\tan(w_j)_{j \geq 1}\right]\right).$$

Let $u_i = \tan(w_i)$; then $u_i = U^\alpha(\mu_i + f_i[(u_j)_{j \geq 1}])$ or $-\Delta u_i + \alpha u_i - f_i[(u_j)_{j \geq 1}] = \mu_i$. Q.E.D.

Appendix: Markov processes. This Appendix should serve as orientation for analysts and probabilists who have little or no acquaintance with Markov processes. There are many sources on the subject to which we could refer the reader, but we mention only [**5**, **8**] and, for the analytically inclined, [**16**], which contains certain aspects of the theory.

First, we introduce the analytic aspects of a Markov process, the semigroup and resolvent. A Lusin topological space E is (homeomorphic to) a Borel subset of a compact metric space $\bar{E}$, and $\mathfrak{E} = \{H \subset E\colon H \text{ is Borel in } \bar{\mathfrak{E}}\}$. Let $(P_t)_{t \geq 0}$ be a Borel measurable sub-Markov semigroup on E. That is,

(A1) For each $x \in E$ and for each $t \geq 0$, $P_t(x, \cdot)$ is a positive measure on $(E, \mathfrak{E})$ of mass less than or equal to 1.

(A2) For each $A \in \mathfrak{E}$ and for each $t \geqslant 0$, $P_t(\cdot, A) \in \mathfrak{E}$.

(A3) $P_{t+s}(x, \cdot) = \int P_t(x, dy)P_s(y, \cdot)$.

If $f \in \mathfrak{E}$, $P_t f(x)$ is defined to be $\int P_t(x, dy)f(y)$. We also assume two regularity hypotheses for P_t.

(A4) $P_0(x, \cdot) = \varepsilon_x(\cdot)$.

(A5) For each $x \in E$ and for each $f \in bC(E)$, $t \to P_t f(x)$ is right continuous.

The resolvent U^α of P_t is defined by setting

$$U^\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x)\, dt$$

and satisfies the resolvent equation $U^\alpha - U^\beta = (\beta - \alpha)U^\alpha U^\beta$.

If E is locally compact with a countable base, then we can give sufficient conditions guaranteeing that P_t is the semigroup of a strong Markov process. Let $C_0(E)$ be the space of continuous functions on E which vanish at infinity. If

(A6) $P_t C_0(E) \subset C_0(E)$, and

(A7) $P_t f \to f$ uniformly on E as $t \to 0$ for each $f \in C_0(E)$,

then there exists a right continuous strong Markov process on E with semigroup P_t [5, p. 46]. However, there are many other ways of constructing Markov processes, such as stochastic differential equations and martingale methods. So for the remainder of this Appendix, we explain what a strong Markov process with semigroup P_t is, assuming it exists.

Let $\Omega = \{w\colon [0, \infty) \to E | w(t)$ is right continuous$\}$, $X_t(w) = w(t)$, $\mathfrak{F}_t^0 = \sigma(X_s\colon s \leqslant t)$, $\mathfrak{F}^0 = \sigma(X_s\colon s \geqslant 0)$. We assume that for each $x \in E$, there is a measure P^x on $(\Omega, \mathfrak{F}^0)$ satisfying

$$P^x\left[X_0 = z, X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n\right]$$
$$= 1_{\{x\}}(z) P_{t_1}\left(1_{A_1} P_{t_2 - t_1}\left(1_{A_2} \cdots P_{t_n - t_{n-1}}\left(1_{A_n}\right)\right) \cdots\right)(x)$$

$(t_1 < t_2 < \cdots < t_n < \infty)$. Then $X = (\Omega, \mathfrak{F}^0, \mathfrak{F}_t^0, X_t, (P^x)_{x \in E})$ is a Markov process or, more precisely, a right continuous simple Markov process. If

(A8) for every $x \in E$, every $\alpha > 0$ and every $f \in b\mathfrak{E}^+$, $P^x[t \to U^\alpha f(X_t)$ is right continuous on $[0, \infty)] = 1$,

then X_t is a strong Markov process. We always assume in this paper that we are dealing with a right continuous strong Markov process. (A8) is equivalent to the strong Markov property. It will not appear explicitly in our work, but it is certainly used to derive some of the results we do apply. We now recall what it is; the uninterested reader should skip to the next paragraph. The space Ω is equipped with translation operators $\theta_t\colon \Omega \to \Omega$ characterized by $X_t(\theta_s w) = X_{t+s}(w)$. Let $T\colon \Omega \to [0, \infty]$ satisfy $\{T \leqslant t\} \in \mathfrak{F}_t^0$: T is called a *stopping time*. Set $\mathfrak{F}_T^0 = \{H \in \mathfrak{F}^0\colon H \cap \{T \leqslant t\} \in \mathfrak{F}_T^0\ \forall t\}$: $\mathfrak{F}_T^0$ is a σ-algebra. If G is $\mathfrak{F}_T^0$-measurable and H is $\mathfrak{F}^0$-measurable, the strong Markov property states that

$$P^x[G \cdot H \circ \theta_T; T < \infty] = P^x\left[G \cdot P^{X(t)}[H]; T < \infty\right].$$

This makes precise the intuitive description that pre-T events and post-T events are conditionally independent given the events at time T.

X is said to be *transient* if there is an $\mathfrak{E}$-measurable function $f > 0$ on E so that $U^0 f < \infty$ on E. Transient processes are discussed in §2; extensions to nontransient processes are given in §4.

We call a function $g \in \mathfrak{E}$ *excessive* for P_t if $P_t g \leqslant g$ and if $\lim_{t \to 0} P_t g = g$. In this case, $g(X_t)$ is right continuous a.s. (P^x) for every x.

This completes our general discussion of Markov processes. The best known is Brownian motion in R^d; where $P_t(x, dy) = p_t(x, y)\, dy$, and,

$$p_t(x, y) = (2\pi t)^{-d/2} \exp(-|y - x|^2/2t).$$

In this special case, X can be constructed to have *continuous* trajectories,

$$\lim_{t \to 0} \left(\frac{P_i - I}{t} \right) = \frac{1}{2} \Delta,$$

and X is transient if and only if $d \geqslant 3$.

In §3, we have used a smattering of stochastic calculus (see **[10]**). An $(\mathfrak{F}_t^0)$-martingale $M_t(w)$ is a stochastic process so that $P^x[M_{t+s} G] = P^x[M_t G]$ for every $G \in \mathfrak{F}_t^0$. We deal only with M_t a Brownian motion. In this case, Ito's change of variables formula holds. Let $A_t(w)$ be a continuous increasing process so that $A_t \in \mathfrak{F}_t^0$. If $f, g \in bC^2(R^d)$, then

$$f(A_t) g(M_t) - f(A_0) g(M_0) = \int_0^t f(A_s) \nabla g(M_s)\, dM_s + \int_0^t f'(A_s) g(M_s)\, dA_s + \frac{1}{2} \int_0^t f(A_s) \Delta g(M_s)\, ds.$$

The first integral on the right above is a stochastic integral, since M is of unbounded variation.

ADDED IN PROOF. H. Brezis has kindly pointed out to us that in Example (2.3) (following (3.13)), the function u must be identically zero. Also, 3 is the dividing line in that example: see the article by H. Brezis in the Séminaire Goulaouic-Meyer-Schwartz 1981–1982. Necessary and sufficient conditions for (1.1) to have a solution have been given by Baras-Pierre and Gallouet-Morel in the case when A is the Laplacian and $f(x) = |x|^{p-1} x$.

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